LIFE LENGTH OF COMPONENTS ESTIMATES WITH BETA-WEIGHTED WEIBULL DISTRIBUTION

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Abstract

Two main distributions are combining by using the logit of beta function by Jones [10]. The weighted Weibull distribution proposed by Shahbaz et al. [19] and beta distribution in order to have a better distribution (beta-weighted Weibull distribution) than each of them individually in terms of the estimate of their characteristics in their parameters. We study and provide a comprehensive treatment of the mathematical properties of the beta weighted Weibull distribution and derive expressions for its moments and moment generating function, survival rate function, hazard rate function, skewness and kurtosis, coefficient of variation and asymptotic behaviours. We also discuss maximum likelihood estimation and provide formulae for the elements of the Fisher information matrix. The new distribution is apply to a lifetime data set and clearly shows that it is much more flexible and has a better representation

of data than weighted Weibull distribution. We hope that this model may attract wider application in biology, biomedical, environmetric, and lifetime data analysis.

1. Introduction

The Weibull distribution has been a powerful probability distribution in reliability analysis; the weighted Weibull distribution is used to adjust the probabilities of the events as observed and recorded; while the beta distribution is one of the skewed distributions used on describing uncertainty or random variation on a system. Patil and Rao [18] investigated how, for instance, truncated distributions and damaged observations can give rise to weighted distribution and Azzalini [3] proposed a model that can be used as an alternative to Gamma and Weibull distribution; and Shahbaz et al. [19] followed the same idea of Azzalini's method to proposed a model that is slightly modifying with additional parameter called "*weighted parameter*". The probability density function of the $ww(\lambda, \beta, \alpha)$ distribution is given by

$$f(x) = \frac{\alpha + 1}{\alpha} \lambda \beta x^{\beta-1} \exp\left(- \lambda x^{\beta}\right) \left[1 - \exp\left(- \alpha \lambda x^{\beta}\right)\right], \ \alpha, \beta, \lambda, x > 0, \quad (1)$$

and the associating cumulative distribution function is given by

$$F_X(x) = \frac{\alpha + 1}{\alpha} \left[\left\{1 - \exp\left(- x^{\beta}\right)\right\} - \frac{1}{\alpha + 1}\left\{1 - \exp\left(- (1 + \alpha)x^{\beta}\right)\right\}\right]. \quad (2)$$

Studies on generalized forms of weighted Weibull distribution are scanty. This paper is arranged as follows: We introduce the new proposed beta weighted Weibull distribution (BWW) including the density and distribution function, the asymptotic behaviours, survival rate function, hazard rate function, etc. and special models these were studies in Section 2. In Section 3, we discussed moment and moment generating function. Section 4 contains the parameter estimation, in Section 5, empirical application to lifetime data set and Section 6 concludes the study.

2. Methods

2.1. The new proposed beta weighted Weibull distribution

Numerous works have been done concerning beta distribution combined with other distributions, in particular, after recent works of Eugene and Famoye [6] and Jones [10], beta generalized logistic (Morais et al. [14]), beta log-logistic (Lemonte [11]), beta-hyperbolic secant (Fischer and Vaughan [8]), beta-Gumbel (Nadarajah and Kotz [15]), moments of the beta Weibull (Cordeiro et al. [4]), beta Weibull (Famoye et al. [7]), beta exponential (Nadarajah and Kotz [16]), beta Pareto (Akinsete et al. [1]), beta Rayleigh (Akinsete and Lowe [2]), beta modified Weibull (Cordeiro et al. [5]), beta exponentiated Pareto (Zea et al. [12]), beta Nakagami (Shittu and Adepoju [20]) beta-beta among others.

Now, let X be a random variable form the distribution with parameters and defined in (1) using the logit of beta function by Jones [10]

$$g(x) = B(a, b)^{-1}[G(x)]^{a-1}[1 - G(x)]^{b-1}g(x). \tag{3}$$

The beta weighted Weibull $BWW(a, b, \lambda, \alpha, \beta)$ distribution is obtained as follows:

$$g_{BWWD}(x) = \frac{1}{B(a, b)}\left[\frac{\alpha + 1}{\alpha}\left\{\left(1 - \ell^{-x^\beta}\right) - \frac{1}{\alpha + 1}\left(1 - \ell^{-(1+\alpha)x^\beta}\right)\right\}\right]^{a-1}$$

$$\times\left[1 - \frac{\alpha + 1}{\alpha}\left\{\left(1 - \ell^{-x^\beta}\right) - \frac{1}{\alpha + 1}\left(1 - \ell^{-(1+\alpha)x^\beta}\right)\right\}\right]^{b-1}$$

$$\times\frac{1 + \alpha}{\alpha}\lambda\beta x^{\beta - 1}\ell^{-\lambda x^\beta}\left(1 - \ell^{-\alpha\lambda x^\beta}\right), \tag{4}$$

where $G(x) = F(x)$ and $g(x) = f(x)$, $a, b, \lambda, \alpha, \beta$ and $x > 0$, $x \sim BWWD(a, b, \lambda, \alpha, \beta)$.

In (4) above, when $b = 1$, it becomes exponentiated weighted Weibull, when $a = 1$, it becomes Lehmann type II weighted Weibull (Badmus et al. [17]) and when $\beta = 1$, the distribution also leads to beta weighted distribution (all are new special sub-models). Then, set $a = b = 1$, it becomes weighted Weibull distribution (parent distribution). Again, assume we set $\lambda = 1$ in (4), we have

$$g_{BWWD}(x) = \frac{1}{B(a,\, b)} \left[\frac{\alpha + 1}{\alpha} \left\{ \left(1 - \ell^{-x^{\beta\beta}} \right) - \frac{1}{\alpha + 1} \left(1 - \ell^{-(1+\alpha)x^{\beta\beta}} \right) \right\} \right]^{a-1}$$

$$\times \left[1 - \frac{\alpha + 1}{\alpha} \left\{ \left(1 - \ell^{-x^{\beta\beta}} \right) - \frac{1}{\alpha + 1} \left(1 - \ell^{-(1+\alpha)x^{\beta\beta}} \right) \right\} \right]^{b-1}$$

$$\times \frac{1 + \alpha}{\alpha} \beta x^{\beta-1} \ell^{-x^\beta} \left(1 - \ell^{-\alpha x^\beta} \right). \tag{5}$$

Such that, $X \sim BWWD\,(\alpha,\, \beta,\, 1,\, a,\, b)$.

From (5), set

$$t(x) = \frac{\alpha + 1}{\alpha} \left\{ \left(1 - \ell^{-x^\beta} \right) - \frac{1}{\alpha + 1} \left(1 - \ell^{-(1+\alpha)x^\beta} \right) \right\}, \tag{6}$$

i.e.,

$$\frac{dt}{dx} = \left(\frac{\alpha + 1}{\alpha} \left\{ \beta x^{\beta-1} \ell^{-x\beta} - \frac{1}{\alpha + 1} (1 + \alpha) \beta x^{\beta-1} \ell^{-(1+\alpha)x^\beta} \right\} \right) \frac{\alpha + 1}{\alpha} \left(1 - \ell^{-x^\beta} \right)$$

$$- \frac{1}{\alpha + 1} \left(1 - \ell^{-(1+\alpha)x^\beta} \right),$$

putting dx into (5), we get

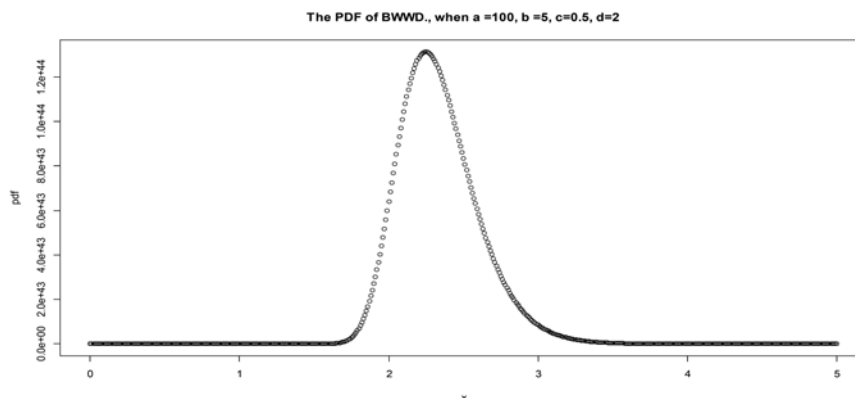$$g_{BWWD}(x) = \frac{1}{B(a,\, b)} \left[\frac{\alpha + 1}{\alpha} \left\{ \left(1 - \ell^{-x^\beta} \right) - \frac{1}{\alpha + 1} \left(1 - \ell^{-(1+\alpha)x^\beta} \right) \right\} \right]^{a-1}$$

$$\times \left[1 - \frac{\alpha + 1}{\alpha} \left\{ \left(1 - \ell^{-x^\beta} \right) - \frac{1}{\alpha + 1} \left(1 - \ell^{-(1+\alpha)x^\beta} \right) \right\} \right]^{b-1} dt, \tag{7}$$

that is,

$$\frac{\partial t}{\partial y} = \frac{\lambda\beta\left(1 + \alpha^\beta\right)y^{\beta-1}e\left(-\lambda y^\beta\right)\left(1 - e^{-\lambda(\alpha y)^\beta}\right)}{\alpha^\beta}.$$

Equation (5) becomes the probability density function of BWW distribution and can be rewritten as

$$g_{BWW}(x) = B(a, b)^{-1}[t]^{a-1}[1 - t]^{b-1}\frac{\partial t}{\partial x}. \tag{8}$$

The PDF of BWWD., when a =100, b =5, c=0.5, d=2

Figure 1. Figure shows the graph of pdf of the BWWD distribution and it is clear that indeed it is rightly skewed and where $c = \alpha$ and $d = \beta$.

2.1.1. Cumulative density function

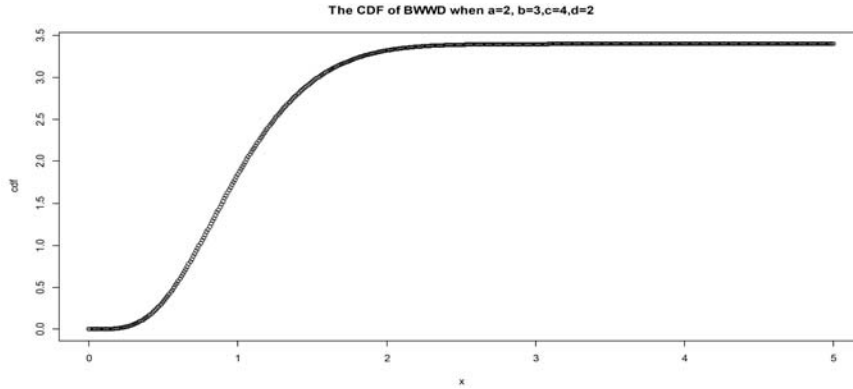The cumulative distribution function (cdf) is deriving and the expression in (7) is given as

$$G_{BWWD}(x) = P(X \le x) = \int_0^x f(t)dt$$

$$= \int_0^x \frac{1}{B(a, b)}\left[\frac{\alpha + 1}{\alpha}\left\{\left(1 - \ell^{-x^\beta}\right) - \frac{1}{\alpha + 1}\left(1 - \ell^{-(1+\alpha)x^\beta}\right)\right\}\right]^{a-1}$$

$$\times\left[1 - \frac{\alpha + 1}{\alpha}\left\{\left(1 - \ell^{-x^\beta}\right) - \frac{1}{\alpha + 1}\left(1 - \ell^{-(1+\alpha)x^\beta}\right)\right\}\right]^{b-1}$$

$$\times\frac{\alpha + 1}{\alpha}\beta x^{\beta-1}\ell^{-x^\beta}\left(1 - \ell^{-\alpha x^\beta}\right)dt, \tag{9}$$

substituting (5) in (8), we obtain

$$G_{BWWD}(x) = P(X \le x) = \int_0^x \frac{1}{B(a,\,b)} T^{a-1}(1-T)^{b-1}\,dt;$$

$$G_{BWWD}(x) = \frac{1}{B(ab)} \int_0^x T^{a-1}(1-T)^{b-1}\,dt = \frac{B(t;\,a,\,b)}{B(a,\,b)}, \qquad (10)$$

where $B(t;\,a,\,b)$ is called an *incomplete beta function.*

Figure 2. This figure also shows the graph of the CDF of the BWW distribution for $a = 2$, $b = 3$, $c = 4$, and $d = 2$.

2.1.2. Survival rate function

The survival rate function of a random variable y with cumulative distribution function $G(y)$ is given by

$$S_{BWW}(x) = 1 - G_{BWW}(x),$$

where $G_{BWW}(x)$ equal to (10), therefore,

$$S_{BWW}(x) = \frac{B(a,\,b) - B(t;\,a,\,b)}{B(a,\,b)}. \qquad (11)$$
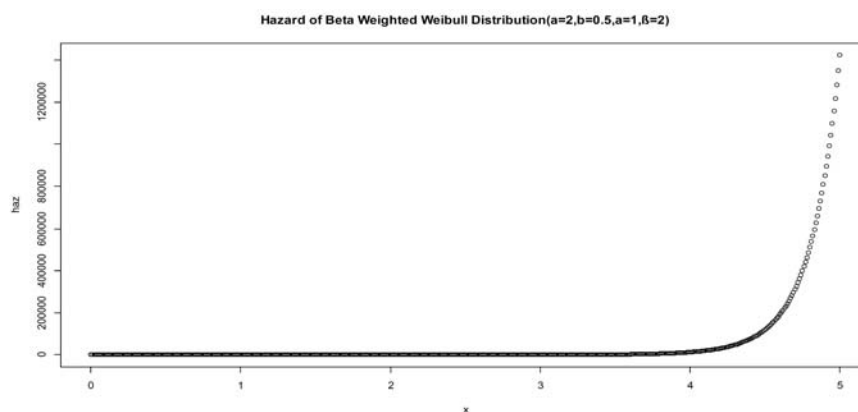
2.1.3. Hazard rate function

$$h_{BWW}(x) = \frac{g_{BWW}(x)}{1 - G_{BWW}(x)},$$

where $g_{BWW}(x) = B(a, b)^{-1} t^{a-1}(1 - t)^{b-1} t'$ and $G_{BWW}(x)$ as in (10).

Substituting $g_{BWW}(x)$ and $G_{BWW}(x)$ in the above expression of hazard function, we obtained the hazard rate function of the BWW distribution as given below:

$$h_{BWW}(x) = \frac{B(a, b)^{-1} t^{a-1}(1 - t)^{b-1} t'}{B(a, b) - B(t; a, b)}, \tag{12}$$

where t is the distribution in (6).

Hazard of Beta Weighted Weibull Distribution(a=2,b=0.5,a=1,ß=2)

Figure 3. The graph of the hazard rate of the beta weighted Weibull distribution for $a = 2$, $b = 0.5$, $\alpha = 1$, and $\beta = 2$ is then shown.

2.1.4. The asymptotic behaviour

The asymptotic properties of the BWW distribution are examined by considering the behaviour of $\lim\limits_{x \to \infty} g_{BWW}(x)$ and $\lim\limits_{x \to 0} g_{BWW}(x)$ as follows:

$$\lim_{x \to 0} g_{BWWD}(x) = \lim_{x \to 0} \frac{1}{B(a, b)} \left[\left(\frac{\alpha + 1}{\alpha} \left\{ \left(1 - \ell^{-x^\beta}\right) - \frac{1}{\alpha + 1}\left(1 - \ell^{(1+\alpha)x^\beta}\right) \right\} \right) \right]^{a-1}$$

$$\times \left[1 - \frac{\alpha + 1}{\alpha} \left\{ \left(1 - \ell^{-x_\beta} \right) - \frac{\alpha + 1}{\alpha} \left(1 - \ell^{-(1+\alpha)x^\beta} \right) \right\} \right]^{b-1}$$

$$\times \frac{\alpha + 1}{\alpha} \beta x^{\beta - 1} \ell^{-x^\beta} \left(1 - \ell^{-\alpha x^\beta} \right).$$

For the sake of simplicity, we take the limit of

$$= \lim_{x \to 0} \frac{\alpha + 1}{\alpha} \beta x^{\beta - 1} \ell^{-x^\beta} \left(1 - \ell^{-\alpha x^\beta} \right) = 0,$$

also

$$= \lim_{x \to \infty} \frac{\alpha + 1}{\alpha} \beta x^{\beta - 1} \ell^{-x^\beta} \left(1 - \ell^{-\alpha x^\beta} \right) = 0.$$

This has shown that at least one mode exists. According to literature, whenever, $x \to \infty$ and $x \to 0$, then the PDF also tends to zero, hence the BWW distribution has mode.

2.2. Special models

2.2.1. Special sub-models

The density (4) is important since it also includes as special sub-models, some distributions not previously focused on or considered in the existing literature. The new special sub-models are given below:

(a) When $\beta = 1$ in (4) the distribution becomes beta weighted exponential distribution (new).

(b) Setting $a = b = \beta = 1$ in (4) reduces to weighted exponential (WE) (Gupta and Kundu [9]) distribution and weighted Weibull (WW) when $a = b = 1$ and $\alpha = \alpha^\beta$ (Mahdy Ramadan [13]).

(c) When $a = 1$, density (5) becomes Lehmann type II weighted Weibull (LWW) (Badmus et al. [17]) distribution (new).

(d) Then, density (5) can also be simplifies to new exponentiated weighted Weibull distribution when $b = 1$ (new) and the WW distribution being the parent distribution (as exemplar) when $a = b = 1$ (Shahbaz et al. [19]).

3. Moments and Moment Generating Function

Hosking [21] described and used in Badmus et al. [17] that when a random variable following a generalized beta generated distribution that is $x \sim GBG(f, a, b, c)$, then $\mu'_r = E[F^{-1} \cup^{\frac{1}{c}}]^r$, where $\cup \sim B(a, b)$, c is a constant and $F^{-1}(x)$ is the inverse of CDF of the weighted Weibull distribution, since $BWW(a, b, \beta, \alpha)$ distribution is a special form when $c = 1$. We then derive the moment generating function (mgf) of the proposed distribution $m(t) = E(e^{tx})$ and the general r-th moment of a beta generated distribution is defined by

$$\mu'_r = \frac{1}{B(a, b)} \int_0^1 [F^{-1}(x)]^r x^{a-1}[1 - x]^{b-1} dy. \tag{13}$$

Cordeiro et al. [5] discussed another mgf of y for generated beta distribution

$$M(t) = \frac{1}{B(a, b)} \sum_{j=0}^{\infty} (-1)^i \binom{b-1}{j} \rho(k, r; aj - 1), \tag{14}$$

where

$$\rho(k, r) = \int_{-\infty}^{\infty} e^{tx} [F(x)]^m f(x) dx.$$

Therefore,

$$M_x^{(t)} = \frac{1}{B(a, b)} \sum_{j=0}^{\infty} (-1)^j \binom{b-1}{j} \int_{-\infty}^{\infty} e^{tx} [F(x)]^{a(j+1)-1} f(x) dx, \tag{15}$$

by substituting both pdf and cdf $(f(x)\,\&\,F(x))$ of the weighted Weibull distribution into (15), we have

$$
M^{(t)}_{BWWD(x)} = \frac{1}{B(a,\,b)} \sum_{j=0}^{\aleph} (-1)^j \binom{b-1}{j} \int \ell^{tx} \left[\left(\frac{\alpha+1}{\alpha} \left\{ \left(1 - \ell^{-x^\beta} \right) \right.\right.\right.
$$

$$
\left.\left.\left. - \frac{1}{\alpha+1} \left(1 - \ell^{-(1+\alpha)x^\beta} \right) \right\} \right) \right]^{a(j+1)-1} x \frac{\alpha+1}{\alpha} \beta x^{\beta-1} \ell^{-x^\beta} \left(1 - \ell^{-\alpha x^\beta} \right),
$$

$$
\tag{16}
$$

setting $a = b = 1$ in (16) gives the moment generating function of the parent distribution.

Now, to obtain the r-th moment of the beta-weighted Weibull distribution, we have the following: Since the moment generating function of weighted Weibull distribution by Shahbaz et al. [19] is given by

$$
M_x(t) = \int_0^\infty \frac{\alpha+1}{\alpha} \beta x^{\beta-1} \ell^{-\alpha x^\beta} \left(1 - \ell^{-\alpha x^\beta} \right) dx
$$

$$
= \sum_{j=0}^\infty \frac{t^j}{j!\,\alpha} \left\{ 1 + \alpha - (1+\alpha)^{-j/\beta} \right\} r\left(1 + \frac{j}{\beta} \right). \tag{17}
$$

Equation (15) can be written as

$$
M_{BWWD(x)}(t) = \frac{1}{B(a,\,b)} \sum_{j=0}^\infty (-1)^i \binom{b-1}{i} \int_0 \left[\frac{\alpha+1}{\alpha} \left\{ \left(1 - \ell^{-x^\beta} \right) \right.\right.
$$

$$
\left.\left. - \frac{1}{\alpha+1} \left(1 - \ell^{-(1+\alpha)x^\beta} \right) \right\} \right]^{a(i+1)-1} \sum_{j=0}^\infty \frac{t^j}{jk} \left\{ 1 + \alpha - (1+\alpha)^{-j/\beta} \right\} \Gamma\left(1 + \frac{j}{\beta} \right),
$$

$$M_{BWWD(x)}(t) = \frac{1}{B(a, b)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^i \binom{b-1}{i} \frac{t^j}{j! \, \alpha} \left\{ 1 + \alpha - (1+\alpha)^{-j/\beta} \right\}$$

$$\times \left(1 + \frac{j}{\beta} \right) \left[\frac{\alpha+1}{\alpha} \left\{ \left(1 - \ell^{-x^\beta} \right) - \frac{1}{\alpha+1} \left(1 - \ell^{-(1+\alpha)x^\beta} \right) \right\} \right]^{a(i+1)-1}.$$

(18)

The r-th moment of BWW distribution can be written from (18) as

$$\mu^1_{BWWD(r)} = E(X^r) = \frac{1}{B(a, b)} \sum_{j=0}^{\infty} (-1)^i \binom{b-1}{i}$$

$$\times \left[\frac{\alpha+1}{\alpha} \left\{ \left(1 - \ell^{-x^\beta} \right) - \frac{1}{\alpha+1} \left(1 - \ell^{-(1+\alpha)x^\beta} \right) \right\} \right]^{a(i+1)-1}$$

$$\times \frac{t^r}{r! \, \alpha} \left\{ 1 + \alpha - (1+\alpha)^{-r/\beta} \right\} \Gamma \left(1 + \frac{r}{\beta} \right).$$

(19)

Putting $a = b = 1$ in (19) leads to the r-th moment of the parent distribution by Shahbaz et al. [19] and is given by

$$\mu^1_r = E(X^r) = \frac{r}{\alpha} \left\{ 1 + \alpha - (1+\alpha)^{-r/\beta} \right\} \Gamma \left(1 + \frac{r}{\beta} \right).$$

It is then easy to obtain the moments and other measures, like the coefficient of variation $(V_{BWWD} a, b, \alpha, \beta)$ of $BWWD(a, b, \alpha, \beta)$, the skewness $S_{BWWD}(a, b, \alpha, \beta)$ and kurtosis $KR_{BWWD}(a, b, \alpha, \beta)$ can also be easily obtained in explicit forms from (19).

4. Parameter Estimation

An attempt is made to obtain the maximum likelihood estimate MLEs of the parameters of the BWW distribution. Now, let θ be a vector of parameters, Cordeiro et al. [5] gave the log-likelihood function for $\theta = (a, b, c, \tau)$, where $\tau = (\alpha, \beta)$, and even was used by Badmus et al. [17].

$$L(\theta) = n \log C - n \log[B(a, b)] + \sum_{i=1}^{n} \log[f(X_1, \tau)]$$

$$+ (a - 1)\sum_{i=1}^{n} \log F(X_1, \tau) + (b - 1)\sum_{i=1}^{n} \log[1 - F^c(X_1, \tau)], \qquad (20)$$

setting $c = 1$, reduces the class of generalized beta distribution to the class of beta generated distribution. Then, we obtain $\theta = (a, b, 1, \tau)$ given as

$$L(\theta) = -n \log[B(a, b)] + \sum_{i=1}^{n} \log[f(X_1, \tau)] + (a - 1)\sum_{i=1}^{n} \log F(X_1, \tau)$$

$$+ (b - 1)\sum_{i=1}^{n} \log\left[1 - F^c(X_1, \tau)\right],$$

where $f(x; \tau)$ and $F(x; \tau)$ as in (1) and (2) above.

$$L_{BWWD}^{(\theta)} = -n \log[B(a, b)] + \sum_{i=1}^{n} \log\left[\frac{\alpha + 1}{\alpha} \beta x^{\beta-1} \ell^{-x^{\beta}}\left(1 - \ell^{-\alpha x^{\beta}}\right)\right]$$

$$+ (\alpha - 1)\left[\sum_{i=1}^{n} \log \frac{\alpha + 1}{\alpha}\left(1 - \ell^{-x^{\beta}}\right) - \frac{1}{\alpha + 1}\left(1 - \ell^{-(1+\alpha)x^{\beta}}\right)\right]$$

$$+ (b - 1)\sum_{i=1}^{n} \log\left[1 - \frac{\alpha + 1}{\alpha}\left[\left(1 - \ell^{-x^{\beta}}\right) - \frac{1}{\alpha + 1}\left(1 - \ell^{-(1+\alpha)x^{\beta}}\right)\right]\right].$$

$$(21)$$

Using differential equation in (21) with respect to (a, b, α, β) and recall that $B(a, b) = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}$

$$\frac{\partial L(\theta)}{\partial a} = -n\frac{\Gamma^1(a)}{\Gamma(a)} + n\frac{\Gamma^1(a+b)}{\Gamma(a+b)} + \sum_{x=1}^{n}\log\frac{\alpha+1}{\alpha}$$

$$\left[\left(1 - \ell^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - \ell^{-(1+\alpha)x^\beta}\right)\right]. \tag{22}$$

$$\frac{\partial L(\theta)}{\partial b} = -n\frac{\Gamma^1(b)}{\Gamma b} + n\frac{\Gamma^1(a+b)}{\Gamma a+b} + \sum_{x=1}^{n}\log$$

$$\left[1 - \frac{1}{\alpha+1}\left[\left(1 - \ell^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - \ell^{-(1+\alpha)x^\beta}\right)\right]\right]. \tag{23}$$

$$\frac{\partial L(\theta)}{\partial \alpha} = \sum_{x=1}^{n}\left[\frac{\frac{\partial}{\partial\alpha}\left(\frac{1}{\alpha+1}\beta x^{\beta-1}\ell^{-x^\beta}\left(1 - \ell^{-\alpha x^\beta}\right)\right)}{\frac{1}{\alpha+1}\beta x^{\beta-1}\ell^{-x^\beta}\left(1 - \ell^{-\alpha x^\beta}\right)}\right]$$

$$+ (a-1)\sum_{x=1}^{n}\frac{\frac{\partial}{\partial\alpha}\left[\frac{1}{\alpha+1}\left(1 - \ell^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - \ell^{-(1+\alpha)x^\beta}\right)\right]}{\frac{\alpha+1}{\alpha}\left[\left(1 - \ell^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - \ell^{-(1+\alpha)x^\beta}\right)\right]}$$

$$+ (b-1)\sum_{x=1}^{n}\frac{\frac{\partial}{\partial\alpha}\left[1 - \frac{\alpha+1}{\alpha}\left(1 - \ell^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - \ell^{-(1+\alpha)x^\beta}\right)\right]}{1 - \frac{\alpha+1}{\alpha}\left[\left(1 - \ell^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - \ell^{-(1+\alpha)x^\beta}\right)\right]}. \tag{24}$$

$$\frac{\partial L(\theta)}{\partial \beta} = \sum_{x=1}^{n}\left[\frac{\frac{\partial}{\partial\beta}\left(\frac{1}{\alpha+1}\beta x^{\beta-1}\ell^{-x^\beta}\left(1 - \ell^{-\alpha x^\beta}\right)\right)}{\frac{\alpha+1}{\alpha}\beta x^{\beta-1}\ell^{-x^\beta}\left(1 - \ell^{-\alpha x^\beta}\right)}\right]$$

$$+ (a-1)\sum_{x=1}^{n}\frac{\frac{\partial}{\partial\beta}\left[\frac{\alpha+1}{\alpha}\left(1 - \ell^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - \ell^{-(1+\alpha)x^\beta}\right)\right]}{\frac{\alpha+1}{\alpha}\left[\left(1 - \ell^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - \ell^{-(1+\alpha)x^\beta}\right)\right]}$$

$$+ (b-1)\sum_{x=1}^{n}\frac{\frac{\partial}{\partial\beta}\left[1 - \frac{\alpha+1}{\alpha}\left(1 - \ell^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - \ell^{-(1+\alpha)x^\beta}\right)\right]}{1 - \frac{\alpha+1}{\alpha}\left[\left(1 - \ell^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - \ell^{-(1+\alpha)x^\beta}\right)\right]}. \tag{25}$$

Equations (22)-(25) can be solved by using Newton Ralphson method to obtain the $\hat{a}$, $\hat{b}$, $\hat{\alpha}$, $\hat{\beta}$ the MLE of (a, b, α, β), respectively.

Taking second derivatives of equations (22), (23), (24), and (25) with respect to the parameters above we can derive the interval estimate and hypothesis tests on the model parameter and inverse of Fisher's information matrix needed.

5. Results and Discussion

3.1. Application to lifetime data

This section, we apply the data set investigated by Shahbaz et al. [19] on life components in years to compare the BWW and WW distribution. The data contains life (grouped data say $(0 - 1.0, 1.0 - 2.0, \ldots, > 5.0)$ and the data set (secondary data) consists of frequency all together 123619 to compare between the results of the proposed BWW, LWW, EWW, and WW. Using R software (codes) to determine some descriptive statistics and the maximum likelihood estimates and the maximized log-likelihood for the beta weighted Weibull (BWW) and weighted Weibull (WW) distributions (with corresponding standard errors in parentheses) are shown in the Tables 1 and 2 below:

Table 1. Descriptive statistics for the life length of components data in years

Min	Q_1	Median	Mean	Q_3	Max	Skewness	Kurtosis
0.000003	0.462500	0.921500	0.959600	1.458000	5.500000	0.1289951	1.926338

Table 2. MLEs of the model parameters and the corresponding standard error

	Estimates and standard errors in parentheses			
Model	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$
BWW	8.95941 (0.89564)	8.17089 (0.80891)	8.36111 (0.84003)	7.91621 (0.78043)
LWW when $a = 1$	-:-	7.90434 (0.68954)	8.44282 (0.73013)	8.70003 (0.75332)
EWW when $b = 1$	7.90434 (0.68954)	-:-	8.44282 (0.73013)	8.70003 (0.75332)
WW When $a = b = 1$	-:-	-:-	8.64992 (0.60540)	8.04152 (0.57077)

Since the values of the estimates are smaller for the BWW distribution compared to other models, therefore, the new model is better representative model to these data.

The asymptotic covariance matrix of the maximum likelihood estimates for the beta weighted Weibull distribution, which is generated from the inverse of Fisher's information matrix and is given by

$$\begin{pmatrix} 0.80216948 & -0.02880174 & 0.00221849 & 0.02674447 \\ -0.02880174 & 0.65433442 & -0.025345533 & -0.02069611 \\ 0.02221849 & -0.02534553 & 0.705642669 & -0.02353514 \\ -0.026744474 & -0.02069611 & -0.02353514 & 0.60907454 \end{pmatrix}.$$

6. Conclusion

The three parameter weighted Weibull distribution later reduced to two parameter by setting $\lambda = 1$ for the sake of simplicity pioneered by Shahbaz et al. [19], is extended by introducing two additional shape parameters called the beta weighted Weibull (BWW) distribution having

a broader class of density functions and hazard rate. This is obtained by taking (2), as baseline cumulative density function of the logit of beta function defined by Jones [10]. We present a detailed study on the mathematical properties of the new propose distribution; and the new model includes as special sub-models the Lehmann weighted Weibull (LWW) (Badmus et al. [17]), weighted exponential (WE) (Gupta and Kundu [9]), weighted Weibull (WW) (Mahdy Ramadan [13]), weighted Weibull (WW) (Shahbaz et al. [19]) and other distribution. We also derive the density and distribution function, survival rate, hazard rate, asymptotic behaviours, moments and moment generating function. The parameters of the propose distribution were estimated and inverse of fisher information matrix is derived. Application to lifetime data set indicates that apart from the beta weighted Weibull is more flexible, it is also has better representation of data and superior to the fit of its principal sub-model. Furthermore, we hope that the new model may be applicable to many areas such as survival analysis, economics, engineering, environmental etc..

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